

CASTELNUOVO-MUMFORD REGULARITY AND SEGRE-VERONESE TRANSFORM

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ABSTRACT. In this paper we give a nice formula for the Castelnuovo-Mumford regularity of the Segre product of modules, under some suitable hypotheses. This extends recent results of David A. Cox, and Evgeny Materov (2009).

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1. INTRODUCTION

Segre and Veronese embeddings of projective variety plays a key role in Algebraic Geometry. From the algebraic point of view K -algebras and their free resolutions are important fields of research. One of the important numerical invariants of S -modules is the Castelnuovo-Mumford regularity. Several approaches to study the Castelnuovo-Mumford regularity of Segre Veronese embeddings was given. The first one was done in [B-M]. S.Goto and K. Watanabe have studied the local cohomology modules of Veronese and Segre transform of graded modules. Motivated by the paper of David A. Cox, and Evgeny Materov (2009), where is computed the Castelnuovo-Mumford regularity of the Segre Veronese embedding, we extend their result and compute the Castelnuovo-Mumford regularity of the Segre product of modules under an hypothesis on the persistence of the local cohomology modules; note that this

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hypothesis is true for Cohen-Macaulay modules. In [G-W] it is given a formula for the local cohomology of a Segre product of modules with depth ≥ 2 . We can easily extend it to modules with depth ≥ 1 . By definition the Castelnuovo-Mumford regularity is the maximum of the "real" regularities of some Segre product of local cohomology modules. In section 2, we define the "virtual" regularities and we prove that even if the "real" regularities and the "virtual" regularities are different, taking the maximum over the set of all "virtual" regularities gives the Castelnuovo-Mumford regularity. This allows us to give nice formulas for Castelnuovo-Mumford regularity of the Segre product of Cohen-Macaulay modules. Some cases of the above results were obtained by studying Hilbert-Poincaré series in [MD1]. In section 3, we study when our hypothesis are true for Segre products of Stanley Reisner rings.

2. SEGRE TRANSFORM, LOCAL COHOMOLOGY AND CASTELNUOVO-MUMFORD REGULARITY

Let S be a polynomial ring over a field K , in a finite number of variables. We suppose that S is graded by the standard graduation $S = \bigoplus_{i \geq 0} S_i$. Let $\mathfrak{m} = \bigoplus_{i \geq 1} S_i$ be the maximal irrelevant ideal. Let M be a finitely generated graded S -module, $M = \bigoplus_{l \geq \sigma} M_l$, with $\sigma \in \mathbb{Z}$ and $M_\sigma \neq 0$.

Remark 2.1. *We choose to take as base ring polynomials rings, because of the paper [G-W], but in fact by using [B-S], all our results will be true over standard Noetherian graded rings $R = R_0[R_1]$, where R_0 is a local ring with infinite residue field. We can assume without loss of generality that the field K is infinite.*

Let M be a finitely generated graded S -module, the local cohomology modules are graded, so we can define $\text{End}(H_{\mathfrak{m}}^i(M)) = \max\{\beta \in \mathbb{Z} \mid (H_{\mathfrak{m}}^i(M))_\beta \neq 0\}$, $r_j(M) = \text{End}(H_{\mathfrak{m}}^j(M)) + j$ and $\text{reg}(M) = \max r_j(M)$ the Castelnuovo-Mumford regularity of M .

Let S_1, S_2 be two polynomial rings on two disjoint sets of variables, for $i = 1, 2$, M_i be a graded S_i -module. The Segre product $M_1 \underline{\otimes} M_2$ is defined by $\bigoplus_{n \in \mathbb{Z}} (M_1)_n \otimes (M_2)_n$. Note that $M_1 \underline{\otimes} M_2$ is a $S_1 \underline{\otimes} S_2$ -module. By using Künneth formula for global cohomology (see [G-W][Proposition (4.1.5)] and Remark 4.1.7, [SV][Section 0.2]), we can extend [G-W][Proposition (4.1.5)]:

Theorem 2.2. *Let S_1, S_2 be two polynomial rings on two disjoint sets of variables, for $i = 1, 2$, M_i be a finitely generated graded S_i -module. Let \mathfrak{m} be the maximal irrelevant ideal of $S_1 \underline{\otimes} S_2$. Assume that $\dim M_1 \geq 1, \dim M_2 \geq 1$, and $\text{depth} M_i \geq \min(2, \dim M_i)$ for $i = 1, 2$. Then for all integers $j, k, l \geq 1$*

$$H_{\mathfrak{m}}^j(M_1 \underline{\otimes} M_2) \simeq (M_1 \underline{\otimes} H_{\mathfrak{m}_2}^j(M_2)) \oplus (H_{\mathfrak{m}_1}^j(M_1) \underline{\otimes} M_2) \bigoplus_{k, l \text{ s.t. } j=k+l-1} (H_{\mathfrak{m}_1}^k(M_1) \underline{\otimes} H_{\mathfrak{m}_2}^l(M_2)).$$

Proof. The case $\dim M_1, \dim M_2 \geq 2$ is [G-W][Proposition (4.1.5)]. Suppose that M_1 is Cohen-Macaulay of dimension 1 and $\text{depth} M_2 \geq 1$. From [G-W][Remark 4.1.7] we get two exact sequences (for the notations we refer to [G-W]):

$$0 \longrightarrow M_1 \longrightarrow M_1^0 \longrightarrow H_{\mathfrak{m}_1}^1(M_1) \longrightarrow, 0$$

$$0 \longrightarrow H_{\mathfrak{m}_1}^1(M_1) \underline{\otimes} M_2 \longrightarrow H_{\mathfrak{m}}^1(M_1 \underline{\otimes} M_2) \longrightarrow M_1^0 \underline{\otimes} H_{\mathfrak{m}_2}^1(M_2) \longrightarrow, 0$$

from the first one we get the exact sequence:

$$0 \longrightarrow M_1 \underline{\otimes} H_{\mathfrak{m}_2}^1(M_2) \longrightarrow M_1^0 \underline{\otimes} H_{\mathfrak{m}_2}^1(M_2) \longrightarrow H_{\mathfrak{m}_1}^1(M_1) \underline{\otimes} H_{\mathfrak{m}_2}^1(M_2) \longrightarrow, 0$$

Our claim follows from these two exact sequences. \square

Note that if M_1, M_2 are Cohen-Macaulay modules of dimension 1, then $M_1 \underline{\otimes} M_2$ is a Cohen-Macaulay module of dimension 1, and $\text{reg}(M_1 \underline{\otimes} M_2) = \max(\text{reg}(M_1), \text{reg}(M_2))$.

Definition 2.3. *From now on, we consider s -polynomial rings (with disjoint set of variables) S_1, \dots, S_s , graded, with irrelevant ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s$. For $i = 1, \dots, s$, let M_i be a finitely generated graded S_i -module such that $\dim M_i \geq 1$ and $\text{depth} M_i \geq \min(2, \dim M_i)$. (Without loss of generality we can assume that at most one module has dimension one). We set the following notations:*

- $d_i = \dim M_i$; $\sigma_i = \min\{l \in \mathbb{Z} : (M_i)_l \neq 0\}$;
 - $r_{i,j} := \text{End}(H_{\mathfrak{m}_i}^j(M_i)) + j$,
 - $C := \{0, \dots, d_1\} \times \{0, \dots, d_2\} \times \dots \times \{0, \dots, d_s\}$,
 - for $u \in C$, $\text{Supp } u = \{i \in \{1, \dots, s\} \mid u_i \neq 0\}$,
 - $E_{i,j} = \begin{cases} H_{\mathfrak{m}_i}^j(M_i) & \text{if } j > 0 \\ M_i & \text{if } j = 0 \end{cases}$;
- for $u \in C$, $E_u = \bigotimes_{i=1}^s E_{i,u_i}$, $\tilde{E}_u = \bigotimes_{i \in \text{Supp } u} E_{i,u_i}$.

We can state the following corollary of Theorem 2.2.

Corollary 2.4. *With the notations introduced in 2.3, for $j \geq 1$, we have*

$$H_{\mathfrak{m}}^j(M_1 \underline{\otimes} \dots \underline{\otimes} M_s) \simeq \bigoplus_{\substack{u \in C, \text{ s.t.} \\ l \in \text{Supp } u}} \sum_{(u_l - 1) = j - 1} E_u,$$

where \mathfrak{m} is the irrelevant maximal ideal of $S_1 \underline{\otimes} \dots \underline{\otimes} S_s$.

Hence we have that

$$\text{reg}(M_1 \underline{\otimes} \dots \underline{\otimes} M_s) = \max_{u \in C} \{1 + \sum_{l \in \text{Supp } u} (u_l - 1) + \text{End}(E_u)\}.$$

In order to compute $\text{reg}(M_1 \underline{\otimes} \dots \underline{\otimes} M_s)$, we have to know when $E_u \neq 0$, and in this case find $\text{End}(E_u)$. In a concrete example, if we know all local cohomology modules, it is possible to compute the regularity of the Segre product, but to give a formula in general is impossible. Our purpose is to give a formula in terms of the regularity data r_{i,u_i} , looking for the optimal hypothesis.

We say that a graded S -module $N = \bigoplus_{l \in \mathbb{Z}} N_l$, not necessarily finitely generated has no gaps if there is no integers $i < j < k$ such that

$$N_i \neq 0; N_k \neq 0; N_j = 0.$$

Example 2.5. Let M be a finitely generated graded S -module with $\text{depth} M \geq 1$, $M = \bigoplus_{l \geq \sigma} M_l$, where $\sigma \in \mathbb{Z}$ and $M_\sigma \neq 0$. Since the field K is infinite, there exists $x \in S_1$, a nonzero divisor of M . The multiplication by x defines an injective map $M_i \rightarrow M_{i+1}$, hence M has no gaps, and for all $l \geq \sigma$, we have $M_l \neq 0$.

Assumption 2.6. From now on, we assume : for any $i = 1, \dots, s$; $\dim M_i \geq 1$ and $\text{depth} M_i \geq \min(2, \dim M_i)$. For any $\text{depth} M_i \leq j \leq \dim M_i$, if $H_{\mathfrak{m}_i}^j(M_i) \neq 0$ then $H_{\mathfrak{m}_i}^j(M_i)$ has no gaps, and $(H_{\mathfrak{m}_i}^j(M_i))_k \neq 0$ for infinitely many k . (Without loss of generality we can assume that at most one module has dimension one).

Note that our assumption is true if all the modules M_i are Cohen-Macaulay. We introduce another piece of notation:

- For $u \in C$, $\Gamma_u = 1 + \sum_{l \in \text{Supp} u} (u_l - 1) + \text{End}(E_u)$.
- For $u \in C$, $\gamma_u = 1 + \sum_{l \in \text{Supp} u} (u_l - 1) + \min_{i \in \text{Supp} u} (\text{End}(H_{\mathfrak{m}_i}^{u_i}(M_i)))$,
- $C_1 := \{u \in C; E_u \neq 0\}$, $C_2 := \{u \in C; \tilde{E}_u \neq 0\}$,

The following Lemma follows immediately from the definitions and the Assumption 2.6.

Lemma 2.7. With the notations introduced in 2.3, and the Assumption 2.6. Let $\epsilon_1, \dots, \epsilon_s$ be the canonical basis of \mathbb{Z}^s . For any $u \in C$:

- (1) $\tilde{E}_u \neq 0$ if and only if $H_{\mathfrak{m}_i}^{u_i}(M_i) \neq 0$ for all $i \in \text{Supp} u$.
- (2) $\text{End} \tilde{E}_u = \min_{i \in \text{Supp} u} (\text{End}(H_{\mathfrak{m}_i}^{u_i}(M_i)))$.
- (3) If $E_u \neq 0$ then $\text{End} E_u = \text{End} \tilde{E}_u$, that is $\Gamma_u = \gamma_u$. If u has full support then $E_u \neq 0$.
- (4) For any $k \notin \text{Supp} u$, $\lambda_k \in \mathbb{N}^*$, $\lambda_k \leq d_k$, such that $H_{\mathfrak{m}_k}^{\lambda_k}(M_k) \neq 0$, we have

$$\gamma_{u+\lambda_k \epsilon_k} = \min(\gamma_u + \lambda_k - 1, \text{End}(H_{\mathfrak{m}_k}^{\lambda_k}(M_k)) + \lambda_k + \sum_{l \in \text{Supp} u} (u_l - 1)).$$

We can state our main result:

Theorem 2.8. *With the notations introduced in 2.3, We have:*

$$\operatorname{reg}(M_1 \underline{\otimes} \dots \underline{\otimes} M_s) \leq \max_{u \in C_2} \{1 + \sum_{l \in \operatorname{Supp} u} (u_l - 1) + \min_{i \in \operatorname{Supp} u} (\operatorname{End}(H_{\mathfrak{m}_i}^{u_i}(M_i)))\}.$$

Moreover the Assumption 2.6 implies the equality.

Proof. Note that

$$\operatorname{reg}(M_1 \underline{\otimes} \dots \underline{\otimes} M_s) = \max(\Gamma_u | E_u \neq 0) \leq \max(\gamma_u | \tilde{E}_u \neq 0),$$

Hence the inequality is trivial. The equality will follows if we prove that

$$\max(\Gamma_u | E_u \neq 0) = \max(\gamma_u | u \in C_2).$$

By the above Lemma if $E_u \neq 0$ then $\Gamma_u = \gamma_u$. We suppose that $E_u = 0$. For any $i = 1, \dots, s$ let δ_i be an integer such that $\operatorname{reg}(M_i) = \operatorname{End}(H_{\mathfrak{m}_i}^{\delta_i}(M_i)) + \delta_i$. We will prove the following statement

(*) There exists $n \notin \operatorname{Supp} u$, such that $\gamma_u \leq \gamma_{u+\delta_n \epsilon_n}$.

If $E_{u+\delta_n \epsilon_n} \neq 0$ then $\gamma_u \leq \gamma_{u+\delta_n \epsilon_n} = \Gamma_{u+\delta_n \epsilon_n}$, otherwise we repeat the argument. This process ends since the local cohomology module $E_{u+\sum_{l \notin \operatorname{Supp} u} \delta_l \epsilon_l}$ is non zero by the Assumption 2.6. Hence there exist some set $J \subset \{1, \dots, n\} \setminus \operatorname{Supp} u$ such that $E_{u+\sum_{n \in J} \delta_n \epsilon_n} \neq 0$ and $\gamma_u \leq \gamma_{u+\sum_{n \in J} \delta_n \epsilon_n} = \Gamma_{u+\sum_{n \in J} \delta_n \epsilon_n}$ the claim is true.

Now we prove (*): we have $E_u = 0 \Leftrightarrow \operatorname{End}(\bigotimes_{i \in \operatorname{Supp} u} H_{\mathfrak{m}_i}^{u_i}(M_i)) < \max_{j \notin \operatorname{Supp} u} \sigma_j$.

Let $n \notin \operatorname{Supp} u$ such that $\max_{j \in \operatorname{Supp} u} \sigma_j = \sigma_n$. Thus the condition $E_u = 0$ is equivalent to

$\min_{i \in \operatorname{Supp} u} (\operatorname{End}(H_{\mathfrak{m}_i}^{u_i}(M_i))) < \sigma_n$. But $\sigma_n \leq \operatorname{reg}(M_n) = \operatorname{End}(H_{\mathfrak{m}_n}^{\delta_n}(M_n)) + \delta_n$ by [B-S, Theorem 15.3.1]. So

$$\min_{i \in \operatorname{Supp} u} (\operatorname{End}(H_{\mathfrak{m}_i}^{u_i}(M_i))) \leq \operatorname{End}(H_{\mathfrak{m}_n}^{\delta_n}(M_n)) + \delta_n - 1.$$

It implies that

$$\gamma_u = 1 + \sum_{l \in \operatorname{Supp} u} (u_l - 1) + \min_{i \in \operatorname{Supp} u} (\operatorname{End}(H_{\mathfrak{m}_i}^{u_i}(M_i))) \leq \operatorname{End}(H_{\mathfrak{m}_n}^{\delta_n}(M_n)) + \delta_n + \sum_{l \in \operatorname{Supp} u} (u_l - 1).$$

On the other hand, since $\operatorname{depth}(M_n) \geq 1$, then $\delta_n \geq 1$, and we have trivially that $\gamma_u \leq \gamma_u + (\delta_n - 1)$ which implies that

$$\gamma_u \leq \min(\gamma_u + \delta_n - 1, \operatorname{End}(H_{\mathfrak{m}_n}^{\delta_n}(M_n)) + \delta_n + \sum_{l \in \operatorname{Supp} u} (u_l - 1)) = \gamma_{u+\delta_n \epsilon_n}.$$

The last equality follows from Lemma 2.7. \square

In order to apply our results to Segre Veronese transform, we recall the following Proposition from [MD2]:

Proposition 2.9. *If M is a Cohen-Macaulay module of dimension d , then $\text{reg } M[\tau]^{<n>} = d - \lceil \frac{d - \text{reg } M + \tau}{n} \rceil$.*

Hence we have the following consequence:

Theorem 2.10. *Let S_1, \dots, S_s be graded polynomial rings on disjoint sets of variables. For all $i = 1, \dots, s$, let M_i be a graded finitely generated S_i -Cohen-Macaulay module with $\dim M_i \geq 1$. (Without loss of generality we can assume that at most one module has dimension one). Let $d_i = \dim M_i$, $b_i = d_i - 1 \geq 1$, $\alpha_i = d_i - \text{reg}(M_i)$, where $\text{reg}(M_i)$ is the Castelnuovo-Mumford regularity of M_i . Then*

$$(1) \text{reg}(M_1 \underline{\otimes} \dots \underline{\otimes} M_s) = \max_{u \in C_2} \{1 + \sum_{l \in \text{Supp } u} b_l - \max_{l \in \text{Supp } u} \{\alpha_l\}\}.$$

(2) For $n_i \in \mathbb{N}$, let $M_i[\tau_i]^{<n_i>}$ be the shifted n_i -Veronese transform of M_i , then

$$\text{reg}(M_1[\tau_1]^{<n_1>} \underline{\otimes} \dots \underline{\otimes} M_s[\tau_s]^{<n_s>}) = \max_{u \in C_2} \{1 + \sum_{l \in \text{Supp } u} b_l - \max_{l \in \text{Supp } u} \{\lceil \frac{\alpha_l + \tau_l}{n_l} \rceil\}\}.$$

As a Corollary we generalize one of the main results of [C-M][Theorem 1.4]

Corollary 2.11. ([C-M][Theorem 1.4]) *For $i = 1, \dots, s$, let S_i be graded polynomial rings on disjoint sets of variables, with $\dim S_i \geq 1$. (Without loss of generality we can assume that at most one ring has dimension one). Let $m_i, n_i \in \mathbb{Z}$, and $S_i[m_i]^{<n_i>}$ be the n_i -Veronese transform of $S_i[m_i]$, then*

$$\text{reg}(S_1[m_1]^{<n_1>} \underline{\otimes} \dots \underline{\otimes} S_s[m_s]^{<n_s>}) = \max_{u \in C_2} \{1 + \sum_{l \in \text{Supp } u} b_l - \max_{l \in \text{Supp } u} \{\lceil \frac{b_l + m_l + 1}{n_l} \rceil\}\}.$$

2.1. Local cohomology Modules without gaps. Let M be a finitely generated graded S -module. We recall the local duality's theorem (see [S]) :

We have an isomorphism :

$$H_{\mathfrak{m}}^i(M) \simeq \text{Hom}_S(\text{Ext}_S^{n-i}(M, S), E(S/\mathfrak{m})).$$

We denote by $D^i(M)$ the finitely generated graded S -module $\text{Ext}_S^{n-i}(M, S)$. The following Lemma follows immediately from the local duality's theorem and the Example 2.5.

Lemma 2.12.

- If $\text{depth}(D^i(M)) \geq 1$ then $H_{\mathfrak{m}}^i(M)$ has no gaps and for all $l \leq \text{End}(H_{\mathfrak{m}}^i(M))$, we have $(H_{\mathfrak{m}}^i(M))_l \neq 0$.
- Let M be a finitely generated graded S -module with $\text{depth } M \geq 1$ and $\dim M = d$. It is known that $\text{depth } D^d(M) \geq \min\{d, 2\}$. Hence the top local cohomology $H_{\mathfrak{m}}^d(M)$ has no gaps and for all $l \leq \text{End}(H_{\mathfrak{m}}^d(M))$, we have $(H_{\mathfrak{m}}^d(M))_l \neq 0$.

- It follows from [M1] that if A is a standard graded simplicial toric ring of dimension d , and $\text{depth} A = d - 1$, then $D^{d-1}(A)$ is a Cohen-Macaulay Module of dimension $d - 2$, so if $d \geq 3$, the module $H_{\mathfrak{m}}^{d-1}(A)$ has no gaps, and for all $l \leq \text{End}(H_{\mathfrak{m}}^{d-1}(M))$, we have $(H_{\mathfrak{m}}^{d-1}(M))_l \neq 0$.

3. SQUARE FREE MONOMIAL IDEALS

Let Δ be a simplicial complex with support on n vertices, labeled by the set $[n] = \{1, \dots, n\}$, $S := K[x_1, \dots, x_n]$ be a polynomial ring, $I_{\Delta} \subset S$ be the Stanley Reisner ideal associated to Δ , that is $I_{\Delta} = (x^F / F \notin \Delta)$. The Alexander dual of Δ is the simplicial complex Δ^* defined by: $F \subset [n]$ is a face of Δ^* if and only if $[n] \setminus F \notin \Delta$. The following theorem is a well known consequence of Hochster's Theorems (see for example [Sb][Proposition 3.8]):

Proposition 3.1. *Let $K[\Delta] := S/I_{\Delta}$ be the Stanley-Reisner ring associated to Δ , $a = (a_1, \dots, a_n) \in \mathbb{Z}$, where for all i , $a_i \leq 0$. Let $F = \text{Supp}(a) \subset [n]$ and $|F| := \text{Card}(F)$ then:*

$$\dim_K(H_{\mathfrak{m}}^i(K[\Delta]))_a = \beta_{i+1-|F|, [n] \setminus F}(K[\Delta^*]).$$

We get the following Corollary:

Corollary 3.2. [Sb][Lemma 3.9]

(1)

$$\dim_K(H_{\mathfrak{m}}^i(K[\Delta]))_0 = \beta_{i+1, n}(K[\Delta^*]),$$

(2) For any integer $j > 0$:

$$\dim_K(H_{\mathfrak{m}}^i(K[\Delta]))_{-j} = \sum_{h=1}^{\min(j, n)} \binom{n}{h} \binom{h+j-1}{j} \beta_{i+1-h, n-h}(K[\Delta^*]).$$

Corollary 3.3. *Let i be an integer such that $(H_{\mathfrak{m}}^i(K[\Delta])) \neq 0$, define k_i as the smallest $0 \leq h \leq n$ such that $\beta_{i+1-h, n-h}(K[\Delta^*]) \neq 0$. If $k_i \geq 1$ then $H_{\mathfrak{m}}^i(K[\Delta])$ has no gaps and $(H_{\mathfrak{m}}^i(K[\Delta]))_k \neq 0$ for all $k \leq \text{end}(H_{\mathfrak{m}}^i(K[\Delta]))$.*

Proof. The formula proved in 3.2 implies that $(H_{\mathfrak{m}}^i(K[\Delta]))_k \neq 0$ for all $k \leq k_i$. The claim is over. \square

Let remark that $\text{depth}(K[\Delta]) \geq 1$ always, so there is a big class of Stanley-Reisner rings that satisfies the Assumption 2.6. In any case if we have s Stanley-Reisner rings, the Castelnuovo-Mumford regularity of their Segre product, can be read off from their Betti's tables, by the Corollary 2.4.

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